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INVARIANT IMBEDDING AND NONVARIATIONAL PRINCIPLES IN ANALYTICAL DYNAMICS

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PREFACE

Part of the Project RAND research program consists of basic supporting studies in mathematics. In this Memorandum the authors provide an integration theory for the canonical equations of motion with parallels to the classical theory of Jacobi. The new approach is applicable to the general case where there is no variational principle underlying the equations of motion.

SUMMARY

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CONTENTS

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1. INTRODUCTION

In several earlier papers [1], [2], we have pointed out that it is possible to associate a one-dimensional steady-state particle transport process with each mechanical process having N degrees of freedom. The generalized displacement vector $q(t)$, $0 \leq t \leq T$, is taken to represent the number of particles of each type passing a point t per unit of time going to the right, and $p(t)$, the generalized momentum, represents the flux to the left at t. Hamilton's equations,

$$(1.1) \qquad \dot{q}_i = H_{p_i},$$

$$(1.2) \qquad -\dot{p}_i = H_{q_i}, \quad 0 \leq t \leq T, \quad i = 1,2,\ldots,N,$$

are the transport equations for this process. The boundary conditions

$$(1.3) \qquad p(T) = c,$$

$$(1.4) \qquad q(0) = w,$$

correspond to incident streams from the right and left. Consider the quantities

$$(1.5) \qquad r(c,T,w) = q(T),$$

$$(1.6) \qquad \tau(c,T,w) = p(0),$$

the reflection and transmission functions. Using invariant imbedding, or other techniques, it is an easy matter to derive partial differential equations for the functions r and τ [3]. These functions can be made the basis for an integration theory of the system of equations (1.1) and (1.2). This was explained in [2].

In this paper, we extend the earlier result in much the same manner that Jacobi extended Hamilton's integration theory [4]. The important point is that the approach presented here applies in the general case where there may not be a variational principle underlying equations (1.1) and (1.2).

2. THE FORMALISM

For simplicity, consider the system of dynamical equations

$$(2.1) \qquad \frac{du}{dt} = F(u,v),$$

$$(2.2) \qquad -\frac{dv}{dt} = G(u,v), \quad 0 \leq t \leq T,$$

along with the boundary conditions

$$(2.3) \qquad u(0) = w, \quad v(T) = c.$$

In earlier notes, we showed the reflection function

$$(2.4) \qquad r(c,T,w) = u(T),$$

and the transmission function

$$(2.5) \qquad \tau(c,T,w) = v(0),$$

satisfy the first-order partial differential equations

$$(2.6) \qquad r_T = F(r,c) + G(r,c)r_c,$$

$$(2.7) \qquad \tau_T = \qquad G(r,c)\tau_c.$$

In addition, they satisfy the initial conditions

$$(2.8) \qquad r(c,0) = w,$$

$$(2.9) \qquad \tau(c,0) = c.$$

Then, on physical grounds, it is clear that

$$(2.10) \qquad r(v,t,w) = u,$$

$$(2.11) \qquad \tau(v,t,w) = \text{const.} = v(0).$$

These represent extensions of Chandrasekhar's principles of invariance [5] to the case of nonlinear transport equations. These equations implicitly give u and v as functions of time and two constants.

We may, however, go further, in the manner of Jacobi. Let

$$(2.12) \qquad r = r(c,T,\alpha)$$

be a solution of equation (2.6) for arbitrary values of the constant α, and let $\tau(c,T,\alpha)$ be a solution of equation (2.7), which now involves α by way of r. Then

$$(2.13) \qquad \tau(v,t,\alpha) = \beta,$$

$$(2.14) \qquad r(v,t,\alpha) = u$$

<u>is a system of equations which implicitly define</u> u <u>and</u> v <u>as functions of</u> t, α, <u>and</u> β; <u>and</u> u <u>and</u> v <u>are</u> <u>solutions of equations</u> (2.1) <u>and</u> (2.2).

Let us verify this statement. Upon differentiation with respect to t, equation (2.13) yields

$$(2.15) \qquad \tau_C(v,t,\alpha)\dot{v} + \tau_T(v,t,\alpha) = 0.$$

From equation (2.7), however, we know that

$$(2.16) \qquad \tau_T(v,t,\alpha) = G(u,v)\tau_C(v,t,\alpha),$$

so that

$$(2.17) \qquad \dot{v} = - G(u,v),$$

provided

$$(2.18) \qquad \tau_C \neq 0.$$

Equation (2.17) is one of the desired relations. Upon differentiating equation (2.14) with respect to t we find

$$(2.19) \qquad r_c(v,t,\alpha)\dot{v} + r_T(v,t,\alpha) = \dot{u}$$

or

$$(2.20) \qquad -r_c(v,t,\alpha)G(u,v) + r_T(v,t,\alpha) = \dot{u}.$$

We recall equation (2.6), and see that

$$(2.21) \qquad \dot{u} = F(u,v).$$

This completes the verification.

3. AN EXAMPLE: THE HARMONIC OSCILLATOR

Consider the equations

$$(3.1) \qquad \dot{q} = \frac{p}{m}, \quad q(0) = w,$$

$$(3.2) \qquad -\dot{p} = kq, \; p(T) = c.$$

The equations for the reflection and transmission functions are

$$(3.3) \qquad r_T = \frac{c}{m} + krr_c$$

and

$$(3.4) \qquad \tau_T = kr\tau_c.$$

We can easily find a one-parameter family of solutions of equation (3.3) using the method of separation of variables,

$$(3.5) \qquad r(c,T,\alpha) = c(km)^{-1/2}\tan[(\frac{k}{m})^{1/2}T + \alpha].$$

A solution of equation (3.4) is

$$(3.6) \qquad \tau(c,T,\alpha) = c\;\sec[(\frac{k}{m})^{1/2}T + \alpha].$$

Consequently, the solution of the system of equations (3.1) and (3.2) is

$$\beta = \tau(p,t,\alpha)$$

$$= p \sec[(\frac{k}{m})^{1/2}t + \alpha],$$

and

$$(3.7) \qquad q = r(p,t,\alpha)$$

$$= p(km)^{-1/2}\tan[(\frac{k}{m})^{1/2}t + \alpha].$$

These expressions reduce to

$$(3.8) \qquad p = \beta \cos[(\frac{k}{m})^{1/2}t + \alpha],$$

$$(3.9) \qquad q = \beta(km)^{-1/2}\sin[(\frac{k}{m})^{1/2}t + \alpha],$$

a form of the solution of the equations of the harmonic oscillator.

4. DISCUSSION

It is evident that this approach is applicable to systems with N degrees of freedom, rather than merely one. In addition, the employment of other principles of invariance [5], such as

$$(4.1) \qquad v(t) = r(u(t), T - t, c),$$

leads to still other relations. Only minor changes in the formulas occur if the right-hand sides are functions of t, as well as u and v. In addition, there are immediate applications of perturbation analysis to results of this nature.

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